Nonnegative Matrices with Power Invariant Zero Patterns

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ABSTRACT

If A is a nonsingular M-matrix, the elements of the sequence $\{A^{-k}\}$ all have the same zero pattern. Using the Drazin inverse, we show that a similar zero pattern invariance property holds for a class of matrices which is larger than the generalized M-matrices.

I. INTRODUCTION

A real square matrix A is called an M-matrix [9] if for some nonnegative matrix B, $A=\alpha I-B$, where α exceeds the spectral radius $\rho(B)$ of B. Since $A^{-1}\geqslant 0$, the sequence $S=\{A^{-k}\}$ is nonnegative.

Recently, several authors independently discovered the interesting fact that the elements of the sequence S all have the same zero pattern. That is, for each pair i, j the ij entry in A^{-k}, namely A_{ij}^{-k}, is zero for all k or is positive for all k.

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For example, consider the following M-matrix A and its inverse:

$$A=\begin{bmatrix} 2 & -2 & 0 & 0 \\ 0 & 2 & 0 & -1 \\ -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \tag{1.1}$$

$$A^{-1}=\frac{1}{4}\begin{bmatrix} 2 & 2 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 2 & 4 & 4 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}. \tag{1.2}$$

One may quickly observe that any row in A^{-1} containing a zero (positive) entry in the ith column is orthogonal (not orthogonal) to the ith column. Hence, the pattern of zeros remains unchanged in successive powers of A^{-1}.

In the nonsingular M-matrix case, the "zero pattern invariance property" can easily be deduced using the Neumann expansion and the characterization

$$A^{-k}=\sum_{i=0}^{\infty}\alpha^{-i-k}\binom{i+k-1}{k-1}B^{i} \tag{1.3}$$

for the nonsingular M-matrix $A=\alpha I-B$, where

$$\binom{n}{r}=\frac{n!}{r!(n-r)!}.$$

This discovery was made while proving a recent conjecture of Michael Engel: If A is a nonsingular M-matrix, $(A^{-1}+\alpha I)^{-1}$ is an M-matrix for all $\alpha\geqslant 0$. The conjecture is an immediate corollary to Lemma 2.1 given below.

The zero pattern invariance property has been discovered independently by Lewin and Neumann [6] and Stadelmaier [11]. Rothblum [10] has also discovered the characterization (1.3).

The purpose of this paper is to specifically draw attention to this interesting property of M-matrices and to show that this "zero pattern invariance property" holds for a much larger class of matrices, even larger than the singular M-matrices. The major results of this paper are attributed to Stadelmaier.

The dual problem, namely that of determining sign patterns of matrices A for which $A^{-1}\geqslant 0$, has been considered recently by Johnson, Leighton, and Robinson [5] and Fiedler and Grone [4].

A matrix A is called a (general) *M-matrix* if for some nonnegative matrix B, $A=\alpha I-B$ for some $\alpha \geqslant \rho(B)$. A fairly complete exposition on the properties and applications of M-matrices is contained in [1], [8], and [9].

Let T denote the set of all matrices A which have the property that $(A+sI)^{-1}\geqslant 0$ on some interval $(0,c]$, $c>0$. It is well known [1] that the set of all M-matrices is contained in T.

If $R(A)$ denotes the range of A, the *index* of A is the smallest nonnegative integer k such that $R(A^{k+1})=R(A^k)$. The *Drazin inverse* of A, denoted by A^D, is the unique solution to the three equations $A^{k+1}X=A^k$, $XAX=X$, $XA=AX$, where k is the index of A [2]. The well-known properties of A^D used in this paper can be found in [2].

II. ZERO PATTERN INVARIANCE

In order to generalize the property described in Section I for nonsingular M-matrices to the larger set T, we must consider a generalization of A^{-1} for each A in T. The concept of "weak Drazin inverse" introduced by Campbell and Meyer [3] is the appropriate generalization needed and is defined following Lemma 2.3.

The main result is Theorem 2.4, but several preliminary facts are needed before the desired generalization is presented.

LEMMA 2.1. *Suppose $A\in T$ and $(A+sI)^{-1}\geqslant 0$ for each $s\in(0,c]$. Then*

$$\rho\left[(A+sI)^{-1}\right]\leqslant \frac{1}{s} \qquad \text{for each} \quad s\in(0,c]. \tag{2.1}$$

Proof. Suppose $\rho[(A+sI)^{-1}]=1/r>1/s$. By the Perron-Frobenius theorem, $A+sI$ has a positive eigenvalue r such that $r<s$. Consequently, zero is an eigenvalue of $[A+(s-r)I]$, which contradicts the hypothesis. ■

LEMMA 2.2. *Suppose $A\in T$ and $(A+sI)^{-1}\geqslant 0$ for each $s\in(0,c)$. Suppose k is a positive integer. Then the following statements are equivalent for each pair i,j:*

(a) $(A+sI)^{-k}_{ij}>0$ *for some* $s\in(0,c)$.
(b) $(A+sI)^{-k}_{ij}>0$ *for each* $s\in(0,c)$.
(c) $(A+tI)^{-p}_{ij}>0$ *for all* p *and each* $t\in(0,c)$.

Proof. (a)⇒(b): Let $W_k(s)=(A+sI)^{-k}$. Then

$$W_k'(s)=-k(A+sI)^{-k-1}=-k(A+sI)^{-k}(A+sI)^{-1}\leqslant 0 \tag{2.2}$$

for each $s\in(0,c)$. The individual entries of $(A+sI)^{-k}$ are rational functions of s which are nonincreasing on the interval $(0,c)$ according to (2.2). Consequently, $(A+sI)_{ij}^{-k}>0$ for each $s\in(0,c)$. Moreover, the diagonal entries of $(A+sI)^{-1}$ are clearly nonzero rational functions of s, so that

$$(A+sI)_{ii}^{-k}>0 \qquad \text{for each } i. \tag{2.3}$$

(b)⇒(c): Suppose $t\in(0,c)$ and $s\in(t,c)$. Then the series expansion

$$\begin{aligned}(A+tI)^{-1}&=\left[(A+sI)-(s-t)I\right]^{-1}\\ &=(A+sI)^{-1}\left[I-(s-t)(A+sI)^{-1}\right]^{-1}\\ &=\sum_{k=1}^{\infty}(s-t)^{k-1}(A+sI)^{-k}\end{aligned} \tag{2.4}$$

will converge by Lemma 2.1, since $t<s$ implies $\rho[(s-t)(A+sI)^{-1}]\leqslant(s-t)/s<1$.

If we now raise both sides of (2.4) to the pth power and observe that the terms in the expansion are all nonnegative by hypothesis, we obtain

$$\begin{aligned}(A+tI)^{-p}&=\left[\sum_{k=1}^{\infty}(s-t)^{k-1}(A+sI)^{-k}\right]^{p}\\ &\geqslant\alpha(A+sI)^{-2p-k}\end{aligned} \tag{2.5}$$

for some positive α. Consequently

$$\begin{aligned}(A+tI)_{ij}^{-p}&\geqslant\left[\alpha(A+sI)^{-2p-k}\right]_{ij}\\ &\geqslant\alpha(A+sI)_{ii}^{-p}(A+sI)_{ij}^{-k}(A+sI)_{jj}^{-p}\\ &>0.\end{aligned} \tag{2.6}$$

(c)$\Rightarrow$(a): This is clear. ∎

LEMMA 2.3. *Suppose* $A \in T$ *and* $(A+sI)^{-1} \geqslant 0$ *for each* $s \in (0,c]$. *Then*

$$G(s)=\left[A+s(I-AA^D)\right] \tag{2.7}$$

is nonsingular for each $s \in (0,c]$ *and*

$$[G(s)]^{-1}=\left[A+s(I-AA^D)\right]^{-1} \tag{2.8}$$

$$=(A+sI)^{-1}(I+sA^D) \tag{2.9}$$

$$=A^D+(A+sI)^{-1}(I-AA^D). \tag{2.10}$$

Proof. Let the expression in (2.9) or (2.10) be denoted by X. It is easily shown that $(A+sI)X=I+sA^D$ in each case. Now

$$(A+sI)XG(s)=(I+sA^D)G(s)=A+sI,$$

and so

$$X=[G(s)]^{-1}.$$ ∎

In [3], Campbell and Meyer called X a *weak Drazin inverse* of the matrix A with index k if $XA^{k+1}=A^k$. The matrix in (2.8) is a weak Drazin inverse of $A \in T$ for each s. We shall now consider the zero patterns of $(A+sI)^{-1}$ and $[G(s)]^{-1}$ and, subsequently, $[G(s)]^{-k}$.

THEOREM 2.4. *Suppose* $A \in T$ *and* $(A+sI)^{-1} \geqslant 0$ *for each* $s \in (0,c)$. *Then for some* $c_0>0$ *the following statements are equivalent*:

(a) *There exists a matrix* Q *which is a polynomial function of* A *such that* $Q_{ij} \neq 0$.
(b) $(A+sI)_{ij}^{-1}>0$ *for each* $s \in (0,c)$.
(c) $[A+s(I-AA^D)]_{ij}^{-1}>0$ *for each* $s \in (0,c_0)$.

Proof. (a)$\Rightarrow$(b): Since Q can also be expressed as a polynomial function of $(A+sI)^{-1}$, $Q_{ij} \neq 0$ implies $(A+sI)_{ij}^{-p} \neq 0$ for some p. Since $(A+sI)_{ij}^{-p}>0$ by Lemma 2.2, $(A+sI)_{ij}^{-1}>0$ for each $s \in (0,c)$.

(b)⇒(c): Using (2.9), we note for $t\in(s,c)$

$$\begin{aligned}\left[A+s(I-AA^D)\right]^{-1}&=(A+sI)^{-1}(I+sA^D)\\ &=(A+sI)^{-1}+(A+sI)^{-1}(sA^D)\\ &=(A+tI)^{-1}(A+sI)^{-1}[(A+sI)+(tI-sI)]+(A+sI)^{-1}(sA^D)\\ &=(A+tI)^{-1}+(t-s)(A+tI)^{-1}(A+sI)^{-1}+(A+sI)^{-1}(sA^D)\\ &=(A+tI)^{-1}+(A+sI)^{-1}\left[(t-s)(A+tI)^{-1}+sA^D\right].\end{aligned}\tag{2.11}$$

Now consider $W=[(t-s)(A+tI)^{-1}+sA^D]$. If $A^D_{ij}=0$, then certainly $W_{ij}\geqslant 0$. If $A^D_{ij}\neq 0$, then for some sufficiently small $s>0$, $W_{ij}>0$, since $(A+tI)^{-1}_{ij}>0$. Hence, there exists $c_0>0$ such that for each $s\in(0,c_0)$, $W\geqslant 0$ and hence,

$$\left[A+s(I-AA^D)\right]^{-1}\geqslant(A+tI)^{-1}.\tag{2.12}$$

Since $[A+s(I-AA^D)]^{-1}_{ij}\geqslant(A+tI)^{-1}_{ij}$, (c) is implied by (b).

(c)⇒(a): This follows because $Q=[A+s(I-AA^D)]^{-1}$ is a polynomial in A and $Q_{ij}\neq 0$. ■

Rothblum [10] has independently obtained Theorem 2.4 for M-matrices.

Corollary 2.5. *Suppose $A\in T$ and $(A+sI)^{-1}\geqslant 0$ for each $s\in(0,c)$. Then for some $c_0>0$, $[G(s)]^{-k}\geqslant 0$ for all positive integers k and each $s\in(0,c_0)$.*

Corollary 2.6. *Suppose $A\in T$ and $(A+sI)^{-1}\geqslant 0$ for each $s\in(0,c)$. Then there exists $c_0>0$ such that for each positive integer k and each $s\in(0,c_0)$ the following matrices have the same zero pattern:*

$$(A+sI)^{-k},\tag{2.13}$$

$$\left[A+s(I-AA^D)\right]^{-k}.\tag{2.14}$$

In particular, if A is an M-matrix, the elements of the sequence $\{[A+s(I-AA^D)]^{-k}\}$ all have the same zero pattern.

COROLLARY 2.7 (Markham [7]). *If A is a nonsingular M-matrix and A^k is upper triangular for some k, then A is upper triangular.*

The following result is a corollary to Lemma 2.1 and is the generalization of Engel's conjecture alluded to earlier.

COROLLARY 2.8. *If A is an M-matrix, then for each $s \geqslant 0$, $B = A(I+sA)^{-1}$ is an M-matrix. In particular, if A is a nonsingular M-matrix, $B = A(I+sA)^{-1} = (A^{-1}+sI)^{-1}$ is an M-matrix.*

This result has been further strengthened by C. R. Johnson [12], replacing sI by an arbitrary nonnegative diagonal matrix D.

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